Tameness in QFTs

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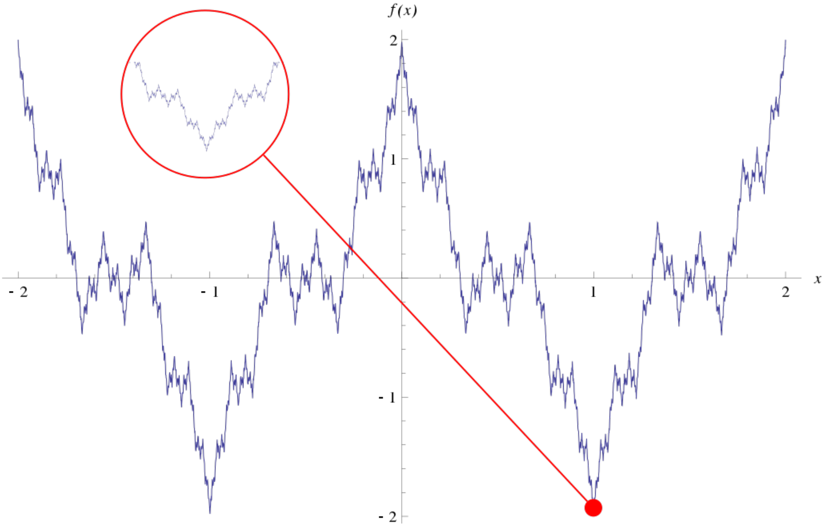
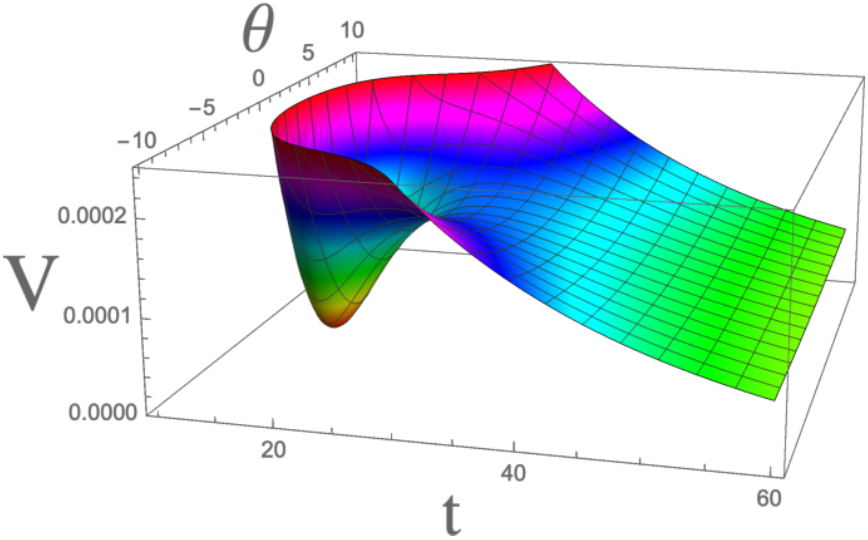
Content

- What is tameness?
- Tameness of perturbative QFT
- Tameness in non-perturbative QFTs
- Consequences of tameness

Math is wild...



Physics is more "tame"



Can this be formalized and used?

- Tame topology/geometry
- Excludes these pathological examples
- Allow only functions which are definable in an o-minimal structure S

Definition of a Structure

Collections $S = (S_n)_{\geq 1}$ *of sets in* $\mathbb{R}^n$ *closed under* $\cup, \cap, \times, /$ *and linear projections containing at least all algebraic sets (= zero sets of polynomials).*

- All sets in S are called definable.
- A function with a definable graph is called definable.

Definition of o-minimality

A structure is o-minimal if the definable subsets of $\mathbb{R}$ *are finite unions of intervals and points*

Definable subsets of $\mathbb{R}$

- Only finitely many points and intervals.
- But the intervals can be infinitely long.
- Higher dimensional sets have to project down to these.

What does this mean in practice?

- o-minimal structures forbid anything infinite discrete
- no integers $\mathbb{Z}$
- no sin(x) and cos(x) for $x \in \mathbb{R}$
- no error or gamma functions
- no infinite monodromies/symmetries

Examples of o-minimal structures:

- $\mathbb{R}_{\mathrm{alg}}$: semi-algebraic sets ($P(x) \geq 0$ instead of $P(x) = 0$)
- $\mathbb{R}_{\mathrm{an}}$: restricted analytic functions
- $\mathbb{R}_{\mathrm{exp}}$: real exponential function
- $\mathbb{R}_{\mathrm{an,exp}}$: combination of the two above

Note: For x in a finite interval sin(x) is a restricted analytic function and thus fine!

How to see definability of a function in practice?

- Very hard question!
- Depends on the allowed domain of the function
- Periods are definable in $\mathbb{R}_{\mathrm{an,exp}}$ [Bakker, Klingler, Tsimerman 20']

$\rightarrow$ Map the problem to a period expression

Tameness in perturbative QFT

- All amplitudes can be expressed in terms of Feynman integrals
- Idea: Map Feynman integrals to periods

Feynman integrals

- Can write a l-loop Feynman integral in a d-dimensional theory in Lee-Pomeransky representation [Lee,Pomeransky 13']

$$I = \frac{\Gamma(\frac{d}{2})}{\Gamma\left(\frac{(\ell+1)d}{2} - v\right)\prod_{j=1}^{n}\Gamma(v_j)} \int_{x_j \geq 0} \prod_{j=1}^{n} \mathrm{d}x_j x_j^{v_j - 1} G^{-\frac{d}{2}} \,. \qquad (1)$$

- G is the Lee-Pomeranski polynomial depending on the masses and external momenta.
- We can interpret this as the defining polynomial of variety in projective space.

$$\omega_i = \int_{\gamma_i} \Omega \equiv \int_{\gamma_i} \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n}{P(a_j, x_i)} \,, \qquad (2)$$

- Lots of technical details involved in the identification! (integration contours, open vs closed chains, divergences ...). In a detailed analysis these work out.

Tameness of perturbative QFT

- Feynman integrals are periods

$\rightarrow$ Feynman integrals are definable

$\rightarrow$ Amplitudes are definable

- If the Lagrangian is tame the perturbative corrections will not destroy this tameness!
- The tameness of the Lagrangian is the tameness conjecture [Grimm 20']

$\rightarrow$ perturbative QFTs are tame if the Lagrangian is tame

What about non-perturbative effects?

- Instantons appear to produce cos potentials$\rightarrow$ appear to be dangerous.
- The Feynman diagram argument does not help due to the non-perturbative nature.
- But: Tameness is not conserved under power series expansion!

$$x^2 = \frac{\pi^2}{3} - 4cos(x) + cos(2x) + \dots \tag{3}$$

- Look at some examples of exactly solvable theories.

Gauged linear sigma models

- 2d superconformal theory with $\mathcal{N} = 2$ supersymmetry.
- Exactly solvable by supersymmetric localization.
- The sphere partition function is given in terms of the Kähler potential of the described geometry

$$Z_{S_2} = e^{-K} = \overline{\Pi}\Sigma\Pi \tag{4}$$

- As the partition function is given in terms of periods it is definable!

Solvable 0d QFTs

- Toy models for higher dimensional theories
- Many 0d QFTs are solvable like the Sine-Gordon model or the ϕ^4 theory

$$Z(m,\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\phi \; e^{-\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4} = \sqrt{\frac{3}{\lambda}} e^{\frac{3m^4}{4\lambda}} \; m \, K_{1/4}\left(\frac{3m^4}{4\lambda}\right) \; , \quad (5)$$

$$Z(g) = \int_{-\pi}^{\pi} \mathrm{d}\phi \; e^{-g \; sin(\phi)^2} = 2e^{-g/2}\pi I_0(g/2) \; , \quad (6)$$

- Modified Bessel functions behave like $\approx \frac{e^{-x}}{\sqrt{x}}$ for $x >> 1$

Consequences of Tameness

- Tameness replaces compactness in mathematical theorems.
- Restricts allowed functions.
- When combined with additional properties tameness becomes much stronger, e.g. tameness $+$ analyticity $\to$ algebraic
- Induces bounds on the transcendence degree of Feynman integrals.
- Relations to decidability
- ... probably much more!

Summary

- Feynman diagrams are periods.
- Assuming the tameness conjecture perturbative QFTs are tame.
- Examples of solvable non-perturbative QFTs are also tame.